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Finite-dimensional reductions of the discrete Toda chain

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Abstract
The problem of construction of integrable boundary conditions for the discrete Toda chain is considered. The restricted chains for properly chosen closure conditions are reduced to the well-known discrete Painlevé equations dP_{III}, dP_V, dP_{VI}. Lax representations for these discrete Painlevé equations are found.

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1. Introduction

In the paper [1] the problem of construction of integrable boundary conditions for the Toda lattice equation

$$q_{n,xx} = \mathrm{e}^{q_{n+1}-q_n} - \mathrm{e}^{q_n-q_{n-1}} \tag{1}$$

is considered. It is shown that generalized Toda equations corresponding to the non-exceptional affine Lie algebras of finite growth arise as finite-dimensional reductions of the Toda chain. Deformations of the boundary conditions found are presented which leads to the Painlevé equations P_{III}, P_V and P_{VI}. Note that the relationship between Painlevé equations P_I–P_{VI} and the Toda chain (1) is established in the framework of the functional approach also [2, 3].

It is well known that the Toda lattice (1) admits several different integrable discretizations [4, 5]. In this paper we will show that all results of the work [1] can be obtained in the case of the following discrete version of the Toda lattice [5]:

$$q_{m+1,n} - 2q_{m,n} + q_{m-1,n} = \ln \frac{\mathrm{e}^{q_{m,n+1}-q_{m,n}} + 1}{\mathrm{e}^{q_{m,n}-q_{m,n-1}} + 1}$$

which can also be presented in variables $u_{m,n} = \mathrm{e}^{q_{m,n}}$

$$u_{m+1,n} = \frac{u_{m,n}^2(1 + u_{m,n+1}/u_{m,n})}{u_{m-1,n}(1 + u_{m,n}/u_{m,n-1})}. \tag{2}$$

The discrete Toda chain (2) is referred to as a two-dimensional reduction of Hirota's bilinear equation [6, 7], which has applications in statistical physics and quantum field theory [8, 9].

There are different possibilities for truncating the chains that conserve the integrability property [10, 5]. In the paper [1] boundary conditions consistent with the higher symmetries of the Toda chain are considered. For chains that admit a zero curvature representation, there is an alternative method for seeking cut-off constraints (boundary conditions) compatible with the conservation laws of the chain [11–13]. In this paper we will apply this method to the discrete Toda chain (2).

The discrete Toda chain (2) is equivalent to the matrix equation

$$L_{m+1,n}(\lambda)A_{m,n}(\lambda) = A_{m,n+1}(\lambda)L_{m,n}(\lambda), \tag{3}$$

which is a consistency condition (the zero curvature equation) of two linear equations

$$Y_{m,n+1}(\lambda) = L_{m,n}(\lambda)Y_{m,n}(\lambda), \tag{4}$$

$$Y_{m+1,n}(\lambda) = A_{m,n}(\lambda)Y_{m,n}(\lambda), \tag{5}$$

where λ is a parameter and $L_{m,n}$, $A_{m,n}$ are matrices of the following form [5]:

$$L_{m,n} = \begin{pmatrix} \lambda + \frac{u_{m,n}}{u_{m-1,n}} & u_{m,n} \\ \lambda \frac{1}{u_{m-1,n}} & 0 \end{pmatrix}, \qquad A_{m,n} = \begin{pmatrix} \lambda & u_{m,n} \\ \lambda \frac{1}{u_{m,n-1}} & -1 \end{pmatrix}.$$

Definition. *We will call a boundary condition*

$$u_{m,0} = F(m, u_{m,1}, u_{m-1,1}, \ldots, u_{m,M}, u_{m-1,M}) \tag{6}$$

compatible with zero curvature equation (3) if equation (5) at the spatial point $n = 1$

$$Y_{m+1,1}(\lambda) = A_{m,1}(\lambda)|_{u_{m,0}=F} Y_{m,1}(\lambda) \tag{7}$$

has an additional point symmetry of the form

$$\tilde{Y}_{m,1}(\tilde{\lambda}) = H(m, [u], \lambda)Y_{m,1}(\lambda), \qquad \tilde{\lambda} = h(\lambda). \tag{8}$$

In other words boundary condition (6) is integrable if there exists a matrix-valued function

$$H(m, [u], \lambda) = H(m, u_{m,1}, u_{m-1,1}, \ldots, u_{m,k}, u_{m-1,k}, \lambda)$$

together with the involution $\tilde{\lambda} = h(\lambda)$ such that for any solution $Y_{m,0}(\lambda)$ of equation (7) the function (8) is a solution of the same equation. It means that the following equality:

$$H(m + 1, [u], \lambda)A_{m,0}(\lambda) = A_{m,0}(\tilde{\lambda})H(m, [u], \lambda) \tag{9}$$

is valid.

We note that equation (9) contains three unknowns (the boundary condition $F(m, [u])$, the involution $\tilde{\lambda}$, and the matrix $H(m, [u], \lambda)$) and generally speaking it has an infinite set of solutions. But if we fix a set of arguments of one of the functions $H(m, [u], \lambda)$ or $F(m, [u])$ (i.e. if we fix number k or M) we obtain additional conditions that suffice to determine the desired functions. In section 2 we represent several kinds of boundary conditions compatible with zero curvature equation (3) of the discrete Toda chain (2). Some of them were found earlier in [5, 11]. In particular, in [5] it is shown that generalized discrete Toda chains corresponding to the all non-exceptional affine Lie algebras can be obtained as finite-dimensional reductions of the discrete Toda equation (2).

The boundary condition (6) reduces the chain (2) to a half-line. To obtain finite-dimensional system we impose two boundary conditions

$$\begin{aligned} u_{m,0} &= F_1(m, u_{m,1}, u_{m-1,1}, \ldots, u_{m,M}, u_{m-1,M}), \\ u_{m,N+1} &= F_2(m, u_{m,1}, u_{m-1,1}, \ldots, u_{m,K}, u_{m-1,K}), \end{aligned} \tag{10}$$

$1 \leqslant M, K \leqslant N$, which are assumed to be compatible with zero curvature equation (3). According to our definition above equality (9) holds at the points $n_1 = 1$ and $n_2 = N + 1$, while the functions $H(m, [u], \lambda)$ and $\tilde{\lambda}$ at these points are equal to the matrices $H_1 = H_1(m, [u], \lambda)$, $H_2 = H_2(m, [u], \lambda)$ and involutions $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$, respectively.

Let us suppose that involutions $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ coincide (i.e. $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}$). In this case we can construct the generating function for the integrals of motion of this system in the standard way [10]

$$g(\lambda) = \operatorname{trace}\bigl(P(m, \lambda) H_1^{-1}(m, \lambda) P^{-1}(m, \tilde{\lambda}) H_2(m, \lambda)\bigr), \tag{11}$$

where $P(m, \lambda) = L_{m,N}(\lambda), \ldots, L_{m,1}(\lambda)$. Similar to the continuous case [14] one can solve this system by utilizing a definite number of symmetries in addition to integrals of motion (see [13]). The set of necessary symmetries can be found by using the properly chosen master symmetries.

The case $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$ in the continuous limit corresponds to finite-dimensional reductions of differential–difference Toda chain (2) obtained by imposing deforming integrable boundary conditions. As mentioned above deformation of the boundary conditions for the Toda chain (1) leads to the Painlevé-type equations (see [1]). It is shown in section 3 that if $N = 1$ and $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$ then the restricted discrete chains for certain choices of closure conditions are reduced to the well-known discrete Painlevé equations dP_{III}, dP_V, dP_{VI} (see (59), (61), (62) below).

2. Boundary conditions consistent with zero curvature equation

In this section we consider boundary condition of the form (6) for the discrete Toda chain (2) assuming that it is compatible with the zero curvature equation. Let us start with the matrix-equation (9) which is the main equation for defining boundary conditions. It gives rise to a system of four scalar equations on elements of matrices $H(m, \lambda)$ and $H(m + 1, \lambda)$ (we denote $h_{ij} = (H(m, \lambda))_{ij}$ and $\bar{h}_{ij} = (H(m + 1, \lambda))_{ij}$)

$$\lambda \bar{h}_{11} + \lambda \bar{h}_{12} F = \tilde{\lambda} h_{11} + h_{21} u_{m,1}, \tag{12}$$

$$\bar{h}_{11} u_{m,1} - \bar{h}_{12} = \tilde{\lambda} h_{12} + h_{22} u_{m,1}, \tag{13}$$

$$\lambda \bar{h}_{21} + \lambda \bar{h}_{22} F = \tilde{\lambda} h_{11} F - h_{21}, \tag{14}$$

$$\bar{h}_{21} u_{m,1} - \bar{h}_{22} = \tilde{\lambda} h_{12} F - h_{22}. \tag{15}$$

Proposition 1. *Suppose that the boundary condition (6) for the discrete Toda chain (2) is compatible with zero curvature equation (3) and the corresponding matrix $H = H(m, \lambda)$ depends on temporal variable m and λ only, i.e. it does not depend on the dynamical variables. Then it is read as*

$$F = \frac{1}{u_{m,0}} = \alpha \mu^{-2m} u_{m,1} + \beta \mu^{-m}. \tag{16}$$

Here and below α, β, μ are arbitrary constants.

Remark. We note that boundary condition (16) was previously found in the particular case when $\mu = 1$ and $\alpha_1 = 0$, $\beta_1 = 0$, $\alpha_2 = 2$, $\beta_2 = 0$ and $\alpha_3 = 0$, $\beta_3 = 1$ (see [5]). Suris elaborated an algebraic structure of finite-dimensional reductions of the discrete Toda chain (2) obtained by imposing one of these boundary conditions. In the case $u_{m,0} = \infty$, $u_{m,N+1} = -\infty$ complete integrability of the corresponding system is proved.

The case $\mu = 1$ with arbitrary constants α, β has been investigated in [13]. It was shown that the corresponding finite-dimensional systems are integrated in quadratures.

Proof of proposition 1. Since the elements of the matrix H do not depend on dynamical variables it follows from equation (13) that $h_{12} = (-\tilde{\lambda})^m a$ where $a = \text{const}$ and

$$\bar{h}_{11} = h_{22}. \tag{17}$$

If we assume that $h_{12} \neq 0$ then the boundary condition F is easily found from (15)

$$F = \frac{\bar{h}_{21} u_{m,1} - \bar{h}_{22} + h_{22}}{\tilde{\lambda} h_{12}}.$$

Substitution of expressions for h_{12} and F into (12) yields

$$-\lambda \bar{h}_{21} u_{m,1} + \lambda \bar{h}_{22} = \tilde{\lambda} h_{11} + h_{21} u_{m,1}.$$

In virtue of independence of the matrix H on dynamical variables the last equation gives $h_{21} = \left(-\frac{1}{\lambda}\right)^m b$, where $b = \text{const}$, and

$$\bar{h}_{22} = \frac{\tilde{\lambda}}{\lambda} h_{11}. \tag{18}$$

Taking into account expressions (17), (18) and independence of the function F upon the parameter λ we immediately find the boundary condition (16) where notation $\alpha = -b$ and $\mu^2 = 1/a$ are used. The matrix H and involution $\tilde{\lambda}$ take the form

$$H(m, \lambda) = \begin{pmatrix} (-1/\lambda)^{m-1} \frac{1}{\lambda+\mu} \beta \mu^{m-1} & (-1/\lambda)^m \mu^{2(m-1)} \\ -(-1/\lambda)^m \alpha & (-1/\lambda)^m \frac{1}{\lambda+\mu} \beta \mu^m \end{pmatrix}, \qquad \tilde{\lambda} = \mu^2/\lambda. \tag{19}$$

The proposition is proved.

Remark. If $F = 0$, i.e. $\alpha = \beta = 0$, then we have

$$H(m, \lambda) = \begin{pmatrix} 0 & (-\lambda)^m \mu^{2(m-1)} \\ 0 & 0 \end{pmatrix}.$$

In this case system (12)–(15) has one more solution

$$H(m, \lambda) = \begin{pmatrix} b & (-\lambda)^m a \\ 0 & b \end{pmatrix}, \qquad \tilde{\lambda} = \lambda.$$

Proposition 2. *Suppose that the boundary condition (6) for the discrete Toda chain (2) compatible with zero curvature equation (3) and the corresponding matrix* $H = H(m, u_{m,1}, u_{m-1,1}, \lambda)$ *depends on dynamical variables* $u_{m,1}$ *and* $u_{m-1,1}$*. Then it is read as*

(1) $$F = \frac{1}{u_{m,0}} = \mu^{-2m} \frac{u_{m,1} u_{m,2}}{u_{m-1,1}} + \frac{(\mu u_{m-1,1} - u_{m,1})^2}{u_{m-1,1}(\mu^{2m} - \mu^2 u_{m,1} u_{m-1,1})} + \frac{\alpha u_{m,1} + \beta\left(\mu^{1-m} u_{m,1}^2 + \mu^m\right)}{\mu^{2m} - \mu^2 u_{m,1} u_{m-1,1}}, \tag{20}$$

(2) $$F = \frac{1}{u_{m,0}} = \frac{u_{m,1} + u_{m,2}}{\alpha u_{m-1,1}^2} - \frac{1}{u_{m,1}}. \tag{21}$$

Remark. Consider the discrete Toda chain (2) with boundary condition of the form (16) where $\alpha = \beta = 0$ at the left endpoint and with (20) where $\mu = 1, \alpha = \beta = 0$ at the right endpoint

$$\mathrm{e}^{-q_{m,0}} = 0, \tag{22}$$

$$q_{m+1,n} - 2q_{m,n} + q_{m-1,n} = \ln \frac{\mathrm{e}^{q_{m,n+1} - q_{m,n}} + 1}{\mathrm{e}^{q_{m,n} - q_{m,n-1}} + 1}, \qquad n = 1, \ldots, N-1, \tag{23}$$

$$q_{m+1,N} - 2q_{m,N} + q_{m-1,N} = \ln \frac{\mathrm{e}^{q_{m-1,N} - 2q_{m,N} - q_{m,N-1}} + \frac{(\mathrm{e}^{q_{m-1,N}} - \mathrm{e}^{q_{m,N}})^2}{\mathrm{e}^{2q_{m,N}}(\mathrm{e}^{q_{m-1,N} + q_{m,N}} - 1)} + 1}{\mathrm{e}^{q_{m,N} - q_{m,N-1}} + 1}. \tag{24}$$

This system in the continuous limit corresponds to the generalized Toda chain

$$\mathrm{e}^{-q_0} = 0, \tag{25}$$

$$q_{n,xx} = \mathrm{e}^{q_{n+1} - q_n} - \mathrm{e}^{q_n - q_{n-1}}, \qquad n = 1, \ldots, N-1, \tag{26}$$

$$\mathrm{e}^{q_{N+1}} = \mathrm{e}^{-q_{N-1}} + \frac{q_{N,x}^2}{2 \sinh q_N}, \tag{27}$$

which is related to the Lie algebras of series D_n [1]. Years ago in [5] the following discrete analogues of (25)–(27) were suggested:

$$\mathrm{e}^{-q_{m,0}} = 0, \tag{28}$$

$$q_{m+1,n} - 2q_{m,n} + q_{m-1,n} = \ln \frac{\mathrm{e}^{q_{m,n+1} - q_{m,n}} + 1}{\mathrm{e}^{q_{m,n} - q_{m,n-1}} + 1}, \qquad n = 1, \ldots, N-2, \tag{29}$$

$$q_{m+1,N-1} - 2q_{m,N-1} + q_{m-1,N-1} = \ln \frac{(\mathrm{e}^{q_{m,N} - q_{m,N-1}} + 1)(\mathrm{e}^{-q_{m,N} - q_{m,N-1}} + 1)}{\mathrm{e}^{q_{m,N-1} - q_{m,N-2}} + 1}, \tag{30}$$

$$q_{m+1,N} - 2q_{m,N} + q_{m-1,N} = \ln \frac{\mathrm{e}^{-q_{m,N} - q_{m,N-1}} + 1}{\mathrm{e}^{q_{m,N} - q_{m,N-1}} + 1}. \tag{31}$$

Unfortunately we failed to find the relation between these two discrete analogues.

Proof of proposition 2. Consider the system of equations (12)–(15). Assume that $h_{12} \neq 0$ then it follows from (15) that

$$F = \frac{\bar{h}_{21} u_{m,1} - \bar{h}_{22} + h_{22}}{\tilde{\lambda} h_{12}}. \tag{32}$$

Let us differentiate equation (13) with respect to the variable $u_{m,2}$. This leads to the equation

$$\frac{\partial(\bar{h}_{11} u_{m,1} - \bar{h}_{12})}{\partial u_{m+1,1}} \frac{\partial u_{m+1,1}}{\partial u_{m,2}} = 0. \tag{33}$$

By setting $\frac{\partial u_{m+1,1}}{\partial u_{m,2}} \neq 0$ (we cannot find F in the opposite case) and integrating (33) one finds that $\bar{h}_{11} u_{m,1} - \bar{h}_{12} = g_1(u_{m,1})$ or $h_{11} = g_1(u_{m-1,1}) + h_{12}/u_{m-1,1}$. Analysis of the left-hand side of equation (13) leads us to the expression $h_{22} = g_1(u_{m,1}) - \tilde{\lambda} h_{12}/u_{m,1}$. Here and below we use $g_i(u_{m,1})$ to denote some functions depending on dynamical variables.

Substituting obtained expressions into (12) yields

$$\lambda\bar{h}_{12}\bar{h}_{21}u_{m,1} - \lambda\bar{h}_{12}\frac{g_1(u_{m+1,1})}{u_{m+1,1}} + \lambda\tilde{\lambda}\bar{h}_{12}^2\frac{1}{u_{m+1,1}} + \lambda\bar{h}_{12}\frac{g_1(u_{m,1})}{u_{m,1}}$$
$$= \tilde{\lambda}^2 h_{12}\frac{g_1(u_{m-1,1})}{u_{m-1,1}} + \tilde{\lambda}^2 h_{12}^2\frac{1}{u_{m-1,1}} + \tilde{\lambda}h_{12}h_{21}u_{m,1} - \lambda\tilde{\lambda}h_{12}\frac{g_1(u_{m,1})}{u_{m,1}}.$$

Recall that $\frac{\partial u_{m+1,1}}{\partial u_{m,2}} \neq 0$, so we can separate the last expression into two equalities

$$\lambda\bar{h}_{12}\bar{h}_{21}u_{m,1} - \lambda\bar{h}_{12}\frac{g_1(u_{m+1,1})}{u_{m+1,1}} + \lambda\tilde{\lambda}\bar{h}_{12}^2\frac{1}{u_{m+1,1}} + \lambda\bar{h}_{12}\frac{g_1(u_{m,1})}{u_{m,1}} = g_2(u_{m,1}), \tag{34}$$

$$\tilde{\lambda}^2 h_{12}\frac{g_1(u_{m-1,1})}{u_{m-1,1}} + \tilde{\lambda}^2 h_{12}^2\frac{1}{u_{m-1,1}} + \tilde{\lambda}h_{12}h_{21}u_{m,1} - \lambda\tilde{\lambda}h_{12}\frac{g_1(u_{m,1})}{u_{m,1}} = g_2(u_{m,1}). \tag{35}$$

Let us find h_{21} from (35)

$$h_{21} = \frac{g_2(u_{m,1})}{\tilde{\lambda}h_{12}u_{m,1}} - \frac{\tilde{\lambda}g_1(u_{m-1,1})}{u_{m,1}u_{m-1,1}} - \frac{\tilde{\lambda}h_{12}}{u_{m,1}u_{m-1,1}} + \frac{\lambda g_1(u_{m,1})}{u_{m,1}^2}.$$

After that equation (34) takes the form

$$\bar{h}_{12}\left(\lambda\frac{g_1(u_{m,1})}{u_{m,1}} - \lambda\tilde{\lambda}\frac{g_1(u_{m,1})}{u_{m+1,1}} + \lambda^2\frac{u_{m,1}g_1(u_{m+1,1})}{u_{m+1,1}^2} - \lambda\frac{g_1(u_{m+1,1})}{u_{m+1,1}}\right)$$
$$= g_2(u_{m,1}) - \frac{\lambda}{\tilde{\lambda}}\frac{g_2(u_{m+1,1})u_{m,1}}{u_{m+1,1}}. \tag{36}$$

One can easily see that equation (14) is rewritten by means of (36) as follows:

$$\left(\frac{g_1(u_{m+1,1})}{u_{m+1,1}\bar{h}_{12}} - \frac{\tilde{\lambda}}{\lambda}\frac{g_1(u_{m-1,1})}{\bar{h}_{12}u_{m-1,1}} + \tilde{\lambda}\frac{h_{12}}{u_{m,1}\bar{h}_{12}} - \tilde{\lambda}\frac{1}{u_{m+1,1}} - \frac{\tilde{\lambda}}{\lambda}\frac{h_{12}}{\bar{h}_{12}u_{m-1,1}} + \frac{1}{u_{m,1}}\right) = 0.$$

The condition $\frac{\partial u_{m+1,1}}{\partial u_{m,2}} \neq 0$ allows one to obtain the following two equalities from the last equation:

$$\frac{g_1(u_{m+1,1})}{u_{m+1,1}} + \bar{h}_{12}\frac{u_{m+1,1} - \tilde{\lambda}u_{m,1}}{u_{m+1,1}u_{m,1}} = g_3(u_{m,1}), \tag{37}$$

$$\frac{\tilde{\lambda}}{\lambda}\frac{g_1(u_{m-1,1})}{u_{m-1,1}} + h_{12}\frac{\tilde{\lambda}}{\lambda}\frac{u_{m,1} - \lambda u_{m-1,1}}{u_{m,1}u_{m-1,1}} = g_3(u_{m,1}). \tag{38}$$

We can find unknown h_{12} from (37)

$$h_{12} = \frac{u_{m,1}u_{m-1,1}}{u_{m,1} - \tilde{\lambda}u_{m-1,1}}\left(g_3(u_{m-1,1}) - \frac{g_1(u_{m,1})}{u_{m,1}}\right). \tag{39}$$

Substitution of this expression for h_{12} into (38) leads to equality

$$\frac{\lambda}{\tilde{\lambda}}g_3(u_{m,1})u_{m,1} + g_1(u_{m,1}) + \tilde{\lambda}g_1(u_{m-1,1}) + \lambda g_3(u_{m-1,1})u_{m-1,1}$$
$$= \lambda u_{m-1,1}g_3(u_{m,1}) + u_{m,1}\frac{g_1(u_{m-1,1})}{u_{m-1,1}} + g_3(u_{m-1,1})u_{m,1} + \lambda u_{m-1,1}\frac{g_1(u_{m,1})}{u_{m,1}}. \tag{40}$$

Differentiation of (40) with respect to the variables $u_{m,1}$ and $u_{m-1,1}$ gives the equation

$$\frac{\partial\left(\frac{g_1(u_{m-1,1})}{u_{m-1,1}} + g_3(u_{m-1,1})\right)}{\partial u_{m-1,1}} = -\lambda\frac{\partial\left(\frac{g_1(u_{m,1})}{u_{m,1}} + g_3(u_{m,1})\right)}{\partial u_{m,1}},$$

from which it follows that

$$g_3(u_{m,1}) = \left(-\frac{1}{\lambda}\right)^{m+1} c_0 u_{m,1} - \frac{g_1(u_{m,1})}{u_{m,1}} + c_1(m).$$

Let $c_i = c_i(\lambda)$ and $c_i(m) = c_i(m, \lambda)$ be some functions depending only on λ and λ, m respectively.

Substituting the expression for $g_3(u_{m,1})$ into (40) yields

$$\begin{aligned}
\frac{1}{\tilde{\lambda}}\left(-\frac{1}{\lambda}\right)^{m} c_0 u_{m,1}^2 + \frac{\lambda}{\tilde{\lambda}} g_1(u_{m,1}) - \frac{\lambda}{\tilde{\lambda}} c_1(m) u_{m,1} - g_1(u_{m,1}) + c_1(m-1) u_{m,1} \\
= -\left(-\frac{1}{\lambda}\right)^{m-1} c_0 u_{m-1,1}^2 - \lambda g_1(u_{m-1,1}) + \lambda c_1(m-1) u_{m-1,1} \\
- \tilde{\lambda} g_1(u_{m-1,1}) + \lambda c_1(m) u_{m-1,1}.
\end{aligned}$$

Left- and right-hand sides of the last equality depend only on $u_{m,1}$ and $u_{m-1,1}$ respectively. Consequently $c_1(m+1) = \frac{\tilde{\lambda}}{\lambda} c_1(m-1)$ and

$$g_1(u_{m,1}) = \frac{1}{\tilde{\lambda} - \lambda}\left((-\tilde{\lambda})^{m+1} c_2 + \left(-\frac{1}{\lambda}\right)^{m} c_0 u_{m,1}^2 - \lambda(c_1(m) - c_1(m+1)) u_{m,1}\right). \tag{41}$$

Return to the equality (36). Taking into consideration (39) and (41) one gets

$$\begin{aligned}
\frac{g_2(u_{m,1})}{u_{m,1}} + \left(-\frac{1}{\lambda}\right)^{m} c_0 g_1(u_{m,1}) + \lambda \frac{g_1^2(u_{m,1})}{u_{m,1}^2} - \lambda c_1(m) \frac{g_1(u_{m,1})}{u_{m,1}} \\
= \frac{\lambda}{\tilde{\lambda}} \frac{g_2(u_{m+1,1})}{u_{m+1,1}} + \left(-\frac{1}{\lambda}\right)^{m+1} \frac{\lambda}{\tilde{\lambda}} c_0 g_1(u_{m+1,1}) + \frac{\lambda^2}{\tilde{\lambda}} \frac{g_1^2(u_{m+1,1})}{u_{m+1,1}^2} \\
- \frac{\lambda^2}{\tilde{\lambda}} c_1(m+1) \frac{g_1(u_{m+1,1})}{u_{m+1,1}}.
\end{aligned}$$

Analysis of the last equation shows that

$$g_2(u_{m,1}) = \left(\frac{\tilde{\lambda}}{\lambda}\right)^{m} c_3 u_{m,1} - \left(-\frac{1}{\lambda}\right)^{m} c_0 g_1(u_{m,1}) u_{m,1} - \lambda \frac{g_1^2(u_{m,1})}{u_{m,1}} + \lambda c_1(m) g_1(u_{m,1}).$$

As the function F does not depend on parameter λ we have $\tilde{\lambda} = \mu^2/\lambda$ and

$$F = \frac{g_2(u_{m,1})}{\mu^2 \bar{h}_{12} h_{12}} - \frac{1}{u_{m,1}}, \tag{42}$$

where

$$h_{12} = \frac{\sqrt{\lambda}}{(\lambda^2 - \mu^2)}\left(-\frac{1}{\lambda}\right)^{m} (a_0 u_{m,1} u_{m-1,1} + \mu^{2m} a_2),$$

$$\begin{aligned}
g_2(u_{m,1}) = \frac{1}{(\lambda^2 - \mu^2)^2}\left(\frac{1}{\lambda}\right)^{2m}\Big(\mu^{2m} a_3 u_{m,1} - \mu^2 a_0^2 u_{m,1}^3 + \mu^{m+1} a_0 a_1 u_{m,1}^2 \\
- \mu^{4m+4} a_2^2 \frac{1}{u_{m,1}} - \mu^{3m+2} a_2 a_1\Big),
\end{aligned}$$

$$g_1(u_{m,1}) = \frac{\sqrt{\lambda}}{(\lambda^2 - \mu^2)}\left(-\frac{1}{\lambda}\right)^{m}\left(\frac{\mu^{2m+2} a_2}{\lambda} - a_0 u_{m,1}^2 + \frac{\mu^m a_1}{\lambda - \mu} u_{m,1}\right),$$

and a_i, $i = 0, 1, 2, 3$ are arbitrary constants.

The function $\bar{h}_{12}$ being contained in the expression for F depends on the variable $u_{m+1,1}$ which is not dynamical, i.e. it can be expressed through variables $u_{m,1}, u_{m-1,1}, u_{m,2}$ and function F

$$u_{m+1,1} = \frac{(u_{m,1} + u_{m,2})u_{m,1}}{u_{m-1,1}(1 + u_{m,1}F)}.$$

Therefore if we denote $a_0 = -a_2\mu^2$, $a_3 = a_2\mu^4(2a_2\mu + \alpha)$, $a_1 = \beta a_2\mu^2$ then the equality (32) gives boundary condition (20). The matrix H and involution $\tilde{\lambda}$ are the following:

$$H = \begin{pmatrix} \frac{g_1(u_{m-1,1})+h_{12}}{u_{m-1,1}} & h_{12} \\ \frac{\lambda g_2(u_{m,1})}{\mu^2 h_{12} u_{m,1}} - \frac{\mu^2(g_1(u_{m-1,1})+h_{12})}{\lambda u_{m,1} u_{m-1,1}} + \lambda \frac{g_1(u_{m,1})}{u_{m,1}^2} & \frac{\lambda g_1(u_{m,1}) - \mu^2 h_{12}}{\lambda u_{m,1}} \end{pmatrix}, \qquad \tilde{\lambda} = \frac{\mu^2}{\lambda}.$$

Now suppose that $h_{12} = 0$. Then the system (12)–(15) takes the form

$$\lambda\bar{h}_{11} = \tilde{\lambda}h_{11} + h_{21}u_{m,1}, \tag{43}$$

$$\bar{h}_{11} = h_{22}, \tag{44}$$

$$\lambda\bar{h}_{21} + \lambda\bar{h}_{22}F = \tilde{\lambda}h_{11}F - h_{21}, \tag{45}$$

$$\bar{h}_{21}u_{m,1} - \bar{h}_{22} = h_{22}. \tag{46}$$

The system (43)–(46) has a nontrivial solution if we assume that $\frac{\partial u_{m+1,1}}{\partial u_{m,2}} = 0$, i.e.

$$F = g_0(u_{m,1}, u_{m-1,1})\left(1 + \frac{u_{m,2}}{u_{m,1}}\right) - \frac{1}{u_{m,1}},$$

and consequently

$$u_{m+1,1} = \frac{u_{m,1}}{u_{m-1,1}g_0(u_{m,1}, u_{m-1,1})}.$$

Taking into account (44) we get from equations (43) and (46)

$$\bar{h}_{11} = h_{11}\frac{\tilde{\lambda}u_{m-1,1} - u_{m,1}}{\lambda u_{m-1,1} - u_{m,1}}, \qquad h_{21} = h_{11}\frac{\tilde{\lambda} - \lambda}{\lambda u_{m-1,1} - u_{m,1}},$$

and so (45) takes the form

$$(\lambda - \tilde{\lambda})(1 + u_{m,1}F)(u_{m+1,1} - \lambda\tilde{\lambda}u_{m-1,1}) = 0. \tag{47}$$

It implies that $u_{m+1,1} = \lambda\tilde{\lambda}u_{m-1,1}$. The other factors in (47) do not vanish. Really, if $\lambda - \tilde{\lambda} = 0$ then H is equal to the identity matrix, which gives no involution. As for the middle factor it coincides with the factor $1 + \frac{u_{m,1}}{u_{m,0}}$ which is contained in the denominator of the chain itself. In the domain of the right-hand side of the chain (2) it must be different from zero. Since $\frac{\partial u_{m+1,1}}{\partial \lambda} = 0$ we have $\tilde{\lambda} = \alpha/\lambda$. Thus, the boundary condition F takes the form (21), the matrix H and the involution $\tilde{\lambda}$, respectively, are of the form

$$H = \begin{pmatrix} g(m) & 0 \\ g(m)\frac{\alpha - \lambda^2}{\lambda(\lambda u_{m-1,1} - u_{m,1})} & g(m+1) \end{pmatrix}, \qquad \tilde{\lambda} = \frac{\alpha}{\lambda},$$

where $g(m) = \prod_{k=0}^{m} \frac{\alpha u_{m-1,1} - \lambda u_{m,1}}{\lambda(\lambda u_{m-1,1} - u_{m,1})}$. The proposition is proved.

3. Discrete Painlevé equations

Consider the truncated system (2), (10) in the case of different involutions $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$ at the endpoints $n = 0$ and $n = 2$. So that the endpoints are taken as close as possible, i.e. $N = 1$. The boundary conditions F_1 and F_2 imposed at $n = 0$ and $n = 2$ are of one of the forms represented by (16) or (20). Denote through $H_1(\lambda, m)$, $H_2(\lambda, m)$ the matrices H corresponding to each endpoint. In the case of (16) and (20) the involutions are of the form $\tilde{\lambda}_1 = \frac{\mu_1^2}{\lambda}$, $\tilde{\lambda}_2 = \frac{\mu_2^2}{\lambda}$. Thus the system (2), (10) takes the form

$$\frac{1}{u_{m,0}} = F_1(m, u_{m,1}, u_{m-1,1}), \tag{48}$$

$$u_{m+1,1} = \frac{u_{m,1}^2(1 + u_{m,2}/u_{m,1})}{u_{m-1,1}(1 + u_{m,1}/u_{m,0})}, \tag{49}$$

$$u_{m,2} = F_2(m, u_{m,1}, u_{m-1,1}). \tag{50}$$

It was shown in [1] that the differential–difference Toda equation (1) admits finite-dimensional reductions of the Painlevé type. The same can be done in our case of purely discrete equations.

Proposition 3. *The system (48)–(50) is equivalent to the matrix equation*

$$A_m(\delta\lambda)M_m(\lambda) = M_{m+1}(\lambda)A_m(\lambda), \tag{51}$$

which is the consistency condition of two linear equations

$$Y_{m+1}(\lambda) = A_m(\lambda)Y_m(\lambda), \tag{52}$$

$$Y_m(\delta\lambda) = M_m(\lambda)Y_m(\lambda), \tag{53}$$

where $M_m(\lambda) = H_1\big(\frac{\mu_2^2}{\lambda}, m\big)L_m^{-1}\big(\frac{\mu_2^2}{\lambda}\big)H_2^{-1}\big(\frac{\mu_2^2}{\lambda}, m\big)L_m(\lambda)$ *and* $\delta = \mu_1^2\big/\mu_2^2$.

Proof. Boundary conditions (48) and (50) are consistent with zero curvature equation (3). It follows from it that equation (5) taken at the spatial points $n = 1$ and $n = 2$

$$Y_{m+1,1}(\lambda) = A_{m,1}(\lambda)Y_{m,1}(\lambda), \qquad Y_{m+1,2}(\lambda) = A_{m,2}(\lambda)Y_{m,2}(\lambda) \tag{54}$$

possesses additional linear transformations

$$Y_{m,1}\left(\frac{\mu_1^2}{\lambda}\right) = H_1(\lambda, m)Y_{m,1}(\lambda), \tag{55}$$

$$Y_{m,2}\left(\frac{\mu_2^2}{\lambda}\right) = H_2(\lambda, m)Y_{m,2}(\lambda). \tag{56}$$

As we said above the system (48)–(50) is equivalent to the consistency condition of equation (54) with the following one:

$$Y_{m,2}(\lambda) = L_{m,1}(\lambda)Y_{m,1}(\lambda). \tag{57}$$

Replacing $\lambda \to \frac{\mu_2^2}{\lambda}$ in (57) and taking into account (56) gives

$$Y_{m,1}\left(\frac{\mu_2^2}{\lambda}\right) = L_{m,1}^{-1}\left(\frac{\mu_2^2}{\lambda}\right)H_2(\lambda, m)L_{m,1}(\lambda)Y_{m,1}(\lambda).$$

Substituting the last expression into (55) we get

$$Y_{m,1}\left(\frac{\mu_1^2}{\lambda}\right) = H_1(\lambda, m)L_{m,1}^{-1}(\lambda)H_2^{-1}(\lambda, m)L_{m,1}\left(\frac{\mu_2^2}{\lambda}\right)Y_{m,1}\left(\frac{\mu_2^2}{\lambda}\right). \tag{58}$$

Replacing again $\lambda \to \frac{\mu_2^2}{\lambda}$ in (58) we get the equality

$$Y_{m,1}\left(\frac{\mu_1^2}{\mu_2^2}\lambda\right) = H_1\left(\frac{\mu_2^2}{\lambda}, m\right)L_{m,1}^{-1}\left(\frac{\mu_2^2}{\lambda}\right)H_2^{-1}\left(\frac{\mu_2^2}{\lambda}, m\right)L_{m,1}(\lambda)Y_{m,1}(\lambda).$$

Omit the second subindex in $u_{m,1}$. So equation (57) is equivalent to equation (53). The proposition is proved.

Thus the system (48)–(50) possesses Lax pair (52), (53), which is typical for the discrete Painlevé equations. Consider several illustrative examples. □

Example 1. The system (48)–(50) with boundary conditions

$$\frac{1}{u_{m,0}} = \alpha_1 u_{m,1} + \beta_1, \qquad u_{m,2} = \alpha_2\mu^{2m}\frac{1}{u_{m,1}} + \beta_2\mu^m$$

gives rise to the equation on variables $u_m = u_{m,1}$

$$u_{m+1}u_{m-1} = \frac{u_m^2 + \beta_2\mu^m u_m + \alpha_2\mu^{2m}}{\alpha_1 u_m^2 + \beta_1 u_m + 1}, \tag{59}$$

which is one of the forms of the third discrete Painlevé equation dP_{III} [15, 16]. Check that in the continuous limit it approaches the P_{III} equation. Return to the variables $u_m = \mathrm{e}^{q_m}$ and take $\mu = \mathrm{e}^{2h}$, $\alpha_1 = \bar{\alpha}_1 h^2$, $\alpha_2 = \bar{\alpha}_2 h^2$, $\beta_1 = \bar{\beta}_1 h^2$, $\beta_2 = \bar{\beta}_2 h^2$. Then equation (59) takes the form

$$q_{m+1} - 2q_m + q_{m-1} = \ln\frac{1 + h^2(\bar{\alpha}_2\,\mathrm{e}^{4mh-2q_m} + \bar{\beta}_2\,\mathrm{e}^{2mh-q_m})}{1 + h^2(\bar{\alpha}_1\,\mathrm{e}^{2q_m} + \bar{\beta}_1\,\mathrm{e}^{q_m})}.$$

Let $h \to 0$ in the last equation, then we have

$$q_{xx} = \bar{\alpha}_2\,\mathrm{e}^{4x-2q} + \bar{\beta}_2\,\mathrm{e}^{2x-q} - \bar{\alpha}_1\,\mathrm{e}^{2q} - \bar{\beta}_1\,\mathrm{e}^{q}. \tag{60}$$

Substitution of $\mathrm{e}^{q(x)} = zy(z)$, $z = \mathrm{e}^x$ in (60) gives the third Painlevé equation [17]

$$y_{zz} = \frac{y_z^2}{y} - \frac{y_z}{z} + \frac{1}{z}(Ay^2 + B) + Cy^3 + \frac{D}{y},$$

where parameters are $A = -\bar{\beta}_1$, $B = \bar{\beta}_2$, $C = -\bar{\alpha}_1$, $D = \bar{\alpha}_2$.

By using proposition 3 we can find a matrix M for zero curvature equation (51) according to equation (59) (we denote $m_{ij} = (M)_{ij}$)

$$m_{12} = \frac{1}{\varphi}(\mu^{m+1}\lambda\beta_2 - \alpha_2(\mu + \lambda)\xi u_m),$$

$$m_{11} = m_{12}\left(\frac{\lambda}{u_m} + \frac{1}{u_{m-1}}\right) + \frac{\mu^m\lambda}{\varphi u_{m-1}}\left(\beta_2\xi - \mu^m(\mu + \lambda)\frac{1}{u_m}\right),$$

$$m_{22} = \frac{1}{\varphi}\left(\beta_1\beta_2\mu^{m+1}\frac{\lambda^2}{\mu^2 + \lambda} - \alpha_2(\mu + \lambda)\eta u_m\right),$$

$$m_{21} = m_{22}\left(\frac{\lambda}{u_m} + \frac{1}{u_{m-1}}\right) + \frac{\mu^m\lambda}{\varphi u_{m-1}}\left(\beta_2\eta - \mu^m(\mu + \lambda)\beta_1\frac{\lambda}{u_m(\mu^2 + \lambda)}\right),$$

where

$$\varphi = \frac{\mu^{2m}}{\mu + \lambda}\left(\alpha_2(\mu + \lambda)^2 - \mu\lambda\beta_2^2\right),$$

$$\xi = \frac{\mu^2\lambda\beta_1 u_{m-1}}{\mu^2 + \lambda} + \frac{\mu^2 u_{m-1} + \lambda u_m}{u_m}, \qquad \eta = \alpha_1\lambda u_{m-1} + \frac{\beta_1\lambda(\mu^2 u_{m-1} + \lambda u_m)}{u_m(\mu^2 + \lambda)}.$$

Example 2. Impose boundary condition (16) at the point $n = 0$

$$\frac{1}{u_{m,0}} = \alpha_1 \mu^{-2m} u_{m,1} + \beta_1 \mu^{-m},$$

and (20) where $\mu = 1$ at the point $n = 2$

$$u_{m,2} = \frac{u_{m-1,1}}{u_{m,1} u_{m,0}} - \frac{(u_{m-1,1} - u_{m,1})^2}{u_{m,1}(1 - u_{m,1} u_{m-1,1})} + \frac{\left(\alpha_2\left(u_{m,1}^2 + 1\right) + \beta_2 u_{m,1}\right) u_{m-1,1}}{u_{m,1}(1 - u_{m,1} u_{m-1,1})}.$$

Under these constraints the Toda chain (2) is reduced to the fifth discrete Painlevé equation dP_V [18]

$$(u_{m+1} u_m - 1)(u_m u_{m-1} - 1) = \frac{pq(u_m - a)(u_m - 1/a)(u_m - b)(u_m - 1/b)}{(u_m - p)(u_m - q)}, \tag{61}$$

where $p = p_0 \mu^m, q = q_0 \mu^m$ and p_0, q_0, a, b are constants such as the following equalities hold

$$p_0 q_0 = \alpha_2, \qquad p_0 + q_0 = -\beta_2,$$
$$a + \frac{1}{a} + b + \frac{1}{b} = \alpha_1, \qquad \left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right) = -(3 + \beta_1).$$

Return to the variables $u_m = e^{q_m}$ in (61) again and take $\mu = e^{-h}$. We use the same constants $\alpha_1, \alpha_2, \beta_1$ and β_2 as in example 1. If $h \to 0$ then we have an equation

$$q_{xx} = \bar{\alpha}_1 e^{2x}(1 - e^{2q}) + \bar{\beta}_1 e^x (e^{-q} - e^q) + \frac{q_x^2}{e^{2q} - 1} + \frac{\bar{\alpha}_2(e^q + e^{-q}) + \bar{\beta}_2}{1 - e^{2q}},$$

which gives the fifth Painlevé equation by substituting $e^{q(x)} = \frac{y(z)+1}{y(z)-1}, z = e^x$ [17]

$$y_{zz} = \left(\frac{1}{2y} + \frac{1}{y-1}\right) y_z^2 - \frac{y_z}{z} + \frac{(y-1)^2}{z^2}\left(Ay + \frac{B}{y}\right) + C\frac{y}{z} + D\frac{y(y+1)}{y-1},$$

where parameters are the following $8A = -\bar{\beta}_2 - 2\bar{\alpha}_2, 8B = \bar{\beta}_2 - 2\bar{\alpha}_2, C = -2\bar{\beta}_1, D = -2\bar{\alpha}_1$.

We will use the following notation:

$$h(\lambda, \mu) = \frac{\sqrt{\lambda}}{\lambda^2 - \mu^2}(\mu^{2m} - \mu^2 u_m u_{m-1}),$$
$$g(u_m, \lambda, \mu, \beta) = \frac{\sqrt{\lambda}}{\lambda^2 - \mu^2}\left(\frac{\mu^{2m+2}}{\lambda} + \mu^2 u_m^2 + \beta \mu^{m+2} \frac{u_m}{\lambda - \mu}\right),$$
$$f(u_m, \lambda, \mu, \alpha, \beta) = \frac{1}{(\lambda^2 - \mu^2)^2}\left(\mu^{2m+4} u_m (2\mu + \alpha) - \mu^6 u_m^3 - \mu^{m+5} \beta u_m^2 - \frac{\mu^{4m+4}}{u_m} - \mu^{3m+4}\beta\right).$$

In this example functions $h, g(u_m)$ and $f(u_m)$ correspond to the following functions $h\left(\frac{1}{\lambda}, 1\right)$, $g\left(u_m, \frac{1}{\lambda}, 1, \beta_1\right)$ and $f\left(u_m, \frac{1}{\lambda}, 1, \alpha_1, \beta_1\right)$. Therefore elements of the matrix M take the form

$$m_{12} = \frac{u_m}{\varphi}(\mu^{2m-2}\lambda \xi_2 u_{m-1} + \eta \zeta),$$
$$m_{11} = m_{12}\left(\frac{\lambda}{u_m} + \frac{1}{u_{m-1}}\right) - \frac{\lambda}{\varphi u_{m-1}}\left(\mu^{2m-2}\lambda h u_m u_{m-1} + \eta \xi_1 \lambda u_m^2\right),$$
$$m_{22} = \frac{u_m}{\varphi}\left(\frac{\lambda^2 \mu^m \beta_1 \xi_2 u_{m-1}}{1 + \lambda\mu} + \frac{\zeta \psi}{h}\right),$$
$$m_{21} = m_{22}\left(\frac{\lambda}{u_m} + \frac{1}{u_{m-1}}\right) - \frac{\lambda}{\varphi u_{m-1}}\left(\lambda \mu^m \beta_1 h u_m u_{m-1} + \psi \xi_1 \lambda u_m^2\right),$$

where

$$\xi_1 = g(u_{m-1}) + h, \qquad \xi_2 = g(u_m) - \lambda h,$$
$$\varphi = \lambda\xi_1 u_m g(u_m) - u_{m-1}(f(u_m)u_m + g(u_m)h),$$
$$\eta = \frac{\lambda\beta_1\mu^{m-1}u_{m-1}}{1+\lambda\mu} + \frac{\mu^{2m-2}(u_{m-1}+\lambda u_m)}{u_m},$$
$$\zeta = f(u_m)u_m u_{m-1} - \lambda^2 h\xi_1 u_m + g(u_m)hu_{m-1},$$
$$\psi = \alpha_1\lambda u_{m-1} + \frac{\lambda\mu^m\beta_1(u_{m-1}+\lambda u_m)}{u_m(1+\lambda\mu)}.$$

Example 3. Consider the chain (2) with boundary conditions (20) where μ is arbitrary constant at the point $n = 0$

$$\frac{1}{u_{m,0}} = \mu^{-2m}\frac{u_{m,1}u_{m,2}}{u_{m-1,1}} + \frac{(\mu u_{m-1,1} - u_{m,1})^2}{u_{m-1,1}(\mu^{2m} - \mu^2 u_{m,1}u_{m-1,1})} + \frac{\alpha_1\left(\mu^{1-m}u_{m,1}^2 + \mu^m\right) + \beta_1 u_{m,1}}{\mu^{2m} - \mu^2 u_{m,1}u_{m-1,1}},$$

and where $\mu = 1$ at the $n = 2$

$$u_{m,2} = \frac{u_{m-1,1}}{u_{m,1}u_{m,0}} - \frac{(u_{m-1,1} - u_{m,1})^2}{u_{m,1}(1 - u_{m,1}u_{m-1,1})} + \frac{\left(\alpha_2\left(u_{m,1}^2 + 1\right) + \beta_2 u_{m,1}\right)u_{m-1,1}}{u_{m,1}(1 - u_{m,1}u_{m-1,1})}.$$

Solving these equations with regard to the variables $u_{m,2}$ and $u_{m,0}$ and substituting them into (2) one gets an equation on variables $u_m = u_{m,1}$ which is the sixth discrete Painlevé equation dP_{VI} [19]

$$\frac{(u_{m+1}u_m - p_{m+1}p_m)(u_m u_{m-1} - p_m p_{m-1})}{(u_{m+1}u_m - 1)(u_m u_{m-1} - 1)} = \frac{(u_m - ap_m)(u_m - p_m/a)(u_m - bp_m)(u_m - p_m/b)}{(u_m - c)(u_m - 1/c)(u_m - d)(u_m - 1/d)}, \tag{62}$$

where $p = p_0\mu^m$, $p_0^2 = 1/\mu$ and a, b, c, d are constants satisfying the following conditions:

$$a + \frac{1}{a} + b + \frac{1}{b} = -\frac{\alpha_1}{\mu p_0}, \qquad \left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right) = 4 + \frac{\beta_1}{\mu},$$
$$c + \frac{1}{c} + d + \frac{1}{d} = \alpha_2, \qquad \left(c + \frac{1}{c}\right)\left(d + \frac{1}{d}\right) = -(4 + \beta_2).$$

One can take $\mu = \mathrm{e}^h$ to get the continuous limit

$$q_{xx} = \frac{\mathrm{e}^{-q} - \mathrm{e}^q}{(1 - \mathrm{e}^{2x})(\mathrm{e}^{q-2x} - \mathrm{e}^{-q})}((q_x - 1)^2 + \bar{\alpha}_1(\mathrm{e}^{q-x} + \mathrm{e}^{x-q}) + \bar{\beta}_1) - \frac{\mathrm{e}^{2x-q} - \mathrm{e}^q}{(1 - \mathrm{e}^{2x})(\mathrm{e}^q - \mathrm{e}^{-q})}\left(q_x^2 - \bar{\alpha}_2(\mathrm{e}^q + \mathrm{e}^{-q}) - \bar{\beta}_2\right).$$

Substituting $\mathrm{e}^{q(x)} = \frac{y(z)+\sqrt{z}}{y(z)-\sqrt{z}}$, $\mathrm{e}^x = \frac{1+\sqrt{z}}{1-\sqrt{z}}$ gives at once the sixth Painlevé equation [17]

$$y_{zz} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)y_z^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)y_z + \frac{y(y-1)(y-z)}{z^2(z-1)^2}\left(A + B\frac{z}{y^2} + C\frac{z-1}{(y-1)^2} + D\frac{z(z-1)}{(y-z)^2}\right),$$

where parameters are the following: $8A = \bar{\beta}_2 + 2\bar{\alpha}_2$, $8B = -\bar{\beta}_2 + 2\bar{\alpha}_2$, $8C = -\bar{\beta}_1 - 2\bar{\alpha}_1$, $8D = \bar{\beta}_1 - 2\bar{\alpha}_1 + 4$.

Elements of the matrix M for equation (62) have the form

$$m_{12} = \frac{u_m}{\varphi}\big(\lambda h_1^2\psi_2 u_{m-1} - \xi_1\zeta\big),$$

$$m_{11} = m_{12}\left(\frac{\lambda}{u_m} + \frac{1}{u_{m-1}}\right) + \frac{\lambda^2 h_1 u_m}{\varphi u_{m-1}}(\xi_1\psi_1 - h_1 h_2 u_{m-1}),$$

$$m_{22} = \frac{u_m}{\varphi}\left(\frac{\lambda h_1 u_{m-1}}{u_m}\psi_2\xi_2 - \eta\zeta\right),$$

$$m_{21} = m_{22}\left(\frac{\lambda}{u_m} + \frac{1}{u_{m-1}}\right) + \frac{\lambda^2 h_1}{\varphi u_{m-1}}(\eta\psi_1 u_m - h_2\xi_2 u_{m-1}),$$

where we denote

$$h_1 = h(1/\lambda, \mu), \qquad h_2 = h(1/\lambda, 1),$$

$$g_1(u_m) = g(u_m, 1/\lambda, \mu, \beta_1), \qquad g_2(u_m) = g(u_m, 1/\lambda, 1, \beta_2),$$

$$f_1(u_m) = f(u_m, 1/\lambda, \mu, \alpha_1, \beta_1), \qquad f_2(u_m) = f(u_m, 1/\lambda, 1, \alpha_2, \beta_2),$$

$$\xi_1 = \lambda u_m g_1(u_{m-1}) - h_1 u_{m-1}, \qquad \xi_2 = g_1(u_m) - \mu^2\lambda h_1,$$

$$\psi_1 = g_2(u_{m-1}) + h_2, \qquad \psi_2 = g_2(u_m) - \lambda h_2,$$

$$\eta = \frac{f_1(u_m)u_{m-1}}{h_1\mu^2} - \lambda g_1(u_m) - \frac{\lambda\mu^2\xi_1}{u_m},$$

$$\zeta = f_2(u_m)u_{m-1} - \lambda^2 h_1\psi_1 + h_1 g_2(u_m)\frac{u_{m-1}}{u_m},$$

$$\varphi = \lambda h_1\psi_1 g_2(u_m)u - h_2 u_{m-1}(f_2(u_m)u_m + g_2(u_m)h_1).$$

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